Compact failure of multiplicativity for linear maps between Banach algebras

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Abstract

The definition of compactness (and that of weak compactness) for a linear map between normed spaces may be extended to multilinear maps in a fairly natural way. We treat compactness as a sort of "smallness" condition for multilinear maps. For Banach algebras A and B we call a linear map, $T : A \to B$, a cf-homomorphism (meaning "compact from a homomorphism") if the bilinear map $S : A \times A \to B$, $S(a, b) = T(a)T(b) - T(ab)$ (i.e. if the "failure to be multiplicative" is a compact bilinear map). We give general theorems showing that such maps are rather well behaved as well as numerous examples. In particular we characterise the pairs of compact, Hausdorff spaces X and Y for which bijective cf-homomorphisms from $C(X)$ to $C(Y)$ are automatically multiplicative.

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Failure of multiplicativity

Definition

Let A and B be algebras and $T : A \to B$ linear. We define the *failure of multiplicativity of* T to be the bilinear map.

$$\begin{aligned} S_T \quad : \quad & A \times A \to B \\ & (a_1, a_2) \mapsto T(a_1)T(a_2) - T(a_1 a_2). \end{aligned}$$

The failure of multiplicativity is studied in questions of *AMNM pairs* (approximately multiplicative maps are near multiplicative maps). This relates to S_T being small (in norm). Specifically:

Definition

Let A and B be Banach algebras. We call (A, B) a *AMNM pair* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $T \in \mathcal{B}(A, B)$ with $\|T\| \leq 1$ and $\|S_T\| < \delta$, there exists a multiplicative map $\Phi \in B(A, B)$ such that $\|T - \Phi\| < \varepsilon$.
We call a Banach algebra A *AMNM* if $(A, \mathbb{C})$ is an ANMN pair.

See Johnson (1986, 1988), Sidney (1997), Jarosz (1997).

We shall consider different notions of "smallness" for S_T, related to compactness.
We need some definitions.

Definition

Let $E_1, \ldots, E_n$ and F be normed spaces. An n-linear map $T : E_1 \times \cdots \times E_n \to F$ is *compact* if the closure of

$$T(\{(x_1, \ldots, x_n) : x_i \in \mathrm{ball}(E_i)\})$$

is compact in the norm topology and *weakly compact* is it is compact in the weak topology.

We denote the set of all [weakly-]compact n-linear maps from $E_1 \times \cdots \times E_n$ to F by $[w]\mathcal{K}^n(E_1, \ldots, E_n; F)$; in the case where $E_1 = \cdots = E_n = E$ we denote $[w]\mathcal{K}^n(E_1, \ldots, E_n; F)$ by $[w]\mathcal{K}^n(E, F)$.
The norm case was considered by Krikorian (1972). He provided the following source of examples.

Example

Let E and F be Banach spaces, U be an open subset of E, $n \in \mathbb{N}$, and $f : U \to F$ be an n times continuously differentiable function which maps bounded sets to relatively compact sets. Then, for $x \in U$ and $k \in \{1, \ldots, n\}$ the kth derivative of f at x is a compact k-linear map from $E \times \cdots \times E$ to F.

Compactness and weak compactness of multilinear maps behave well with respect to the projective tensor product.
Krikorian also showed the following in the norm case and when $n = 2$. We proved the more general version (indeed for a larger class of topologies on F than we state here) independently.

Theorem

Let $E_1, \ldots, E_n$ and F be normed spaces. A multilinear map $T \in \mathcal{B}(E_1, \ldots, E_n; F)$ is [weakly-]compact if and only if the linear map $\widetilde{T} \in \mathcal{B}\left(\widehat{\bigotimes}_{i=1}^{n} E_i, F\right)$ with $\widetilde{T}(x_1 \hat{\otimes} \ldots \hat{\otimes} x_n) = T(x_1, \ldots, x_n)$ is [weakly-]compact.

It follows that $\mathcal{K}^n(E_1, \ldots, E_n; F)$ and $w\mathcal{K}^n(E_1, \ldots, E_n; F)$ are closed subspaces of $\mathcal{B}^n(E_1, \ldots, E_n; F)$.

It is easy to check that compactness and weak compactness of multilinear maps also behave well with respect to composition. For example:

Lemma

Let $E_1, \dots, E_n$, $F_1, \dots, F_n$, G and H be normed spaces. Let $T \in \mathcal{B}^n(F_1, \dots, F_n; G)$, $T' \in \mathcal{B}(G, H)$ and for $i = 1, \dots, n$, let $T_i \in \mathcal{B}(E_i, F_i)$ and define $S \in \mathcal{B}^n(E_1, \dots, E_n; H)$ by

$$S(e_1, \dots, e_n) = T' \circ T(T_1(e_1), \dots, T_n(e_n)).$$

Now assume, either that each T_i is compact or that either of T or T' is compact. Then S is compact.

Compact failure of multiplicativity

Now we combine the concepts disscussed so far.

Definition

Let A and B be normed algebras and let T be a linear map. Let S_T be the failure of multiplicativity of T. We call T a *cf-homomorphism* (where "cf" stands for "compact from") if S_T is compact and a *wcf-homomorphism* (where "wcf" stands for "weakly compact from") if S_T is weakly compact.
If, for each, $a \in A$ we have that $S_T(a, \cdot)$ and $S_T(\cdot, a)$ are [weakly] compact linear maps, we say that T is a *semi-[w]cf-homomorphism*.

Implications between conditions

We have the following relationships between conditions.

- Homomorphisms are cf-homomorphisms.
- Compact linear maps are cf-homomorphisms.
  - Cf-homomorphisms are wcf-homomorphisms.
  - Weakly compact linear maps are wcf-homomorphisms.
    - Wcf-homomorphisms are semi-wcf-homomorphisms.
  - Cf-homomorphisms are semi-cf-homomorphisms.
    - Semi-cf-homomorphisms are semi-wcf-homomorphisms.

Are there any other implications between the conditions?

Examples

Cf-homomorphisms need not be homomorphisms or (even weakly) compact. Indeed there is a bijective, bounded cf-homomorphism between non-isomorphic, infinite dimensional C^*-algebras.

Example

Let $A = \ell^\infty$ with the pointwise product and let B be the vector space $\mathbb{M}_2 \oplus A$ with the product $(\mathbf{A}, a)(\mathbf{B}, b) = (\mathbf{AB}, ab)$ and the unique C^*-algebra norm. We define a linear map $T : A \to B$ by

$$(a_n)_{n\in\mathbb{N}} \to \left(\begin{bmatrix} a_1, & a_2 \\ a_3, & a_4 \end{bmatrix}, (a_{n-4})_{n\in\mathbb{N}} \right).$$

It is easy to check that T is a bounded linear isomorphism and that S_T has finite rank (and so is compact).

Since cf-homomorphisms are "a compact map away from being homomorphisms" one may conjecture that if T_1 is a continuous homomorphism and T_2 is a compact linear map then $T := T_1 + T_2$ must be a cf-homomorphism (compare AMNM). However, it need not even be a semi-wcf-homomorphism.

Example

Let $A = \ell^\infty$. Denote the identity element of A by 1. Let $T_1 : A \to A$ be the identity homomorphism $T_1(a) = a$, $a \in A$ and let $T_2 : A \to A$ be the bounded, rank 1, linear map given by $T_2(a) = a_1 1$. Set $T = T_1 + T_2$. Let $(a_n) \in A$ have $a_1 = 0$ and set (e_{1n}) to have $e_{11} = 1$ and $e_{1k} = 0$ $(k > 1)$. Then,

$$S_T(e_1, a) = T(e_1)T(a) - T(e_1 a) = 1a - T(0) = a.$$

So $S_T(e_1, (\text{ball}(A)))$ contains an isometric copy of $\text{ball}(A)$ and so $S_T(e_1, \cdot)$ is not weakly compact.

Bounded semi-cf-homomorphisms need not be wcf-homomorphisms.

Example

Let $A = c_0$, with pointwise multiplication. Then for each $a \in A$ multiplication by a is a compact map. Thus *every* bounded linear map for A to itself is a semi-cf-homomorphism. Let $T : A \to A$ be given by $T(a) = 2a$; then $\mathrm{ball}(A) \subset S_T\left((\mathrm{ball}(A))^2\right)$. Since the unit ball of c_0 is not relatively compact in the weak topology, it follows that T is not a wcf-homomorphism.

Weakly compact linear maps need not be semi-cf-homomorphisms.

Example

Let A_0 be ℓ^2 with pointwise multiplication, and let A be the one dimensional unitisation of this algebra. Since A is isomorphic as a Banach space to ℓ^2, all linear maps into A are weakly compact. Denote the unit of A by 1 and let $T \in \mathcal{B}(A)$ be given by $T(a) = 2a$ $(a \in A)$. The $S_T(1, \cdot) = T$, which is bijective and so not compact.

Some basic general results

Bounded ?f-homomorphisms behave well with respect to composition.

Theorem

Let A, B and C be Banach algebras and let $T_1 : A \longrightarrow B$ and $T_2 : B \longrightarrow C$ be bounded ?f-homomorphisms. Then $T_2 \circ T_1$ is a bounded ?f-homomorphism.

Proof.

Let $a, b \in A$. Then

$$\begin{aligned} S_{T_2 \circ T_1}(a,b) &= (T_2 \circ T_1)(a)(T_2 \circ T_1)(b) - (T_2 \circ T_1)(ab) \\ &= T_2(T_1(a)T_1(b)) - T_2(T_1(a)T_1(b)) \\ &\quad + (T_2 \circ T_1)(a)(T_2 \circ T_1)(b) - (T_2 \circ T_1)(ab) \\ &= T_2(S_{T_1}(a,b)) - S_{T_2}(T_1(a), T_1(b)). \end{aligned}$$

□

Note, this means that Banach algebras together with bounded ?f-homomorphisms (for some specific choice of "?") form a (concrete) category.

Theorem

Let A and B be Banach algebras and T be a bounded ?f-homomorphism which is bijective. Then the inverse mapping $T^{-1} : B \to A$ is a ?f-homomorphism.

Proof.

By the Banach isomorphism theorem, T^{-1} is a bounded linear map. Also, for $b, b' \in B$ we have,

$$\begin{aligned} S_{T^{-1}}(b, b') &= T^{-1}(b)T^{-1}(b') - T^{-1}(bb') \\ &= T^{-1}(b)T^{-1}(b') - T^{-1}(S_T(T^{-1}(b), T^{-1}(b')) \\ &\quad + T(T^{-1}(b)T^{-1}(b'))) \\ &= -T^{-1} \circ S_T(T^{-1}(b), T^{-1}(b')). \end{aligned}$$

□

Splitting of extensions

We now take a short detour into the Hochschild-Kamowitz (or continuous Hochschild) cohomology of Banach algebras.

Definition

Let A be a Banach algebra. A *Banach extension of A* is a short exact sequence of Banach algebras and bounded algebra homomorphisms,

$$\Sigma(\mathfrak{A}, I) : 0 \to I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} A \to 0$$

such that I is a closed ideal in $\mathfrak{A}$ and ι is the inclusion map. We shall say an extension is *good* if $I^2 = 0$ and $q(\mathfrak{A})$ is closed (N.B. non-standard terminology).

For a good extension $\Sigma(\mathfrak{A}, I)$, we may make I an Banach A-bimodule using the following actions,

$$x \cdot q(a) = xa,\ q(a) \cdot x = ax, \qquad a \in \mathfrak{A}, x \in I$$

Let A be a Banach algebra, E a Banach A-bimodule and $\Sigma(\mathfrak{A}, I)$ a good Banach extension of A. We say that Σ is a *good Banach extension of A by E* if E and I are isomorphic Banach A-bimodules.

Definition

A Banach extension

$$\Sigma : 0 \to I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} A \to 0$$

splits if there is a bounded algebra homomorphism $Q : A \to \mathfrak{A}$, such that $q \circ Q = \mathrm{id}_A$.

Let A be a Banach algebra and E a Banach A-bimodule. Splitting of good Banach extensions of A by E is studied using the second Kamowitz-Hochschild cohomology group of A with coefficients in E. That is

$$\mathcal{H}^2(A; E) = \ker \delta^2 / \mathrm{im} \delta^1,$$

for some complex of Banach algebras and bounded algebra homomorphisms

$$\begin{aligned} \mathcal{B}^\bullet(A, E) : 0 \quad &\to \quad E \xrightarrow{\delta^0} \mathcal{B}(A, E) \xrightarrow{\delta^1} \mathcal{B}^2(A, E) \ldots \\ &\xrightarrow{\delta^{n-1}} \quad \mathcal{B}^n(A, E) \xrightarrow{\delta^n} \mathcal{B}^{n+1}(A, E) \ldots . \end{aligned}$$

There are well-known results such as the following (Badé, Dales and Lykova 1999).

Proposition

Let A be a Banach algebra and E a Banach A-bimodule. The following are equivalent:

(a) *All good Banach extensions of A by E split.*

(b) $\mathcal{H}^2(A; E) = \{0\}$.

We now consider Banach extensions which split in the category of Banach algebras and bounded cf-homomorphisms. That is, a short exact sequence of Banach algebras and bounded homomorphisms,

$$\Sigma : 0 \longrightarrow I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} A \longrightarrow 0,$$

such that there is a bounded cf-homomorphism Q, with $q \circ Q = \mathrm{id}_A$. We say that an extension that splits in this category *cf-splits*.

Let

$$H^2_K(A;E) = \ker\, \delta^2 \cap \mathcal{K}^2(A;E)/\mathrm{im}\delta^1 \cap \mathcal{K}(A;E),$$

We have the following analogue of the previous result.

Theorem

Let A be a Banach algebra and E a Banach A-bimodule. The following are equivalent:

(a) *All good Banach extensions of A by E, which cf-split, split.*

(b) $H^2_K(A;E) = \{0\}$.

We have more general results of this type and all "?f" variants also hold.

Commutative C^*-algebras

Let X be a locally compact Hausdorff space and $C_0(X)$ the commutative C^*-algebra of all continuous functions on X vanishing at infinity. It is standard that all commutative C^*-algebras are of this form.
We say a non-empty topological space, X, is *perfect* if it has no isolated points: i.e. there is no $x \in X$ with $\{x\}$ open. The behaviour of our various types of ?f-homomorphisms between the algebras $C_0(X)$ varies hugely depending on whether X is perfect or not.

Theorem

Let X and Y be non-empty locally compact Hausdorff spaces. Suppose that X is perfect and let $T : C_0(X) \to C_0(Y)$ be a bijective, bounded semi-wcf-homomorphism. Then T is a multiplicative.
Conversely, if we suppose that X is not perfect, then there is a bounded linear isomorphism $T : C_0(X) \to C_0(X)$ such that T is not multiplicative but S_T is one dimensional (so T is a cf-homomorphism).

The second part is easy.
If $X = \{x\}$ set $T(x) = 2x$.
Otherwise, let $x_0 \in X$ be an isolated point and $x_1 \in X \setminus \{x_0\}$ and for all $f \in C_0(X)$ and $x \in X$ set

$$T(f)(x) = \begin{cases} f(x_0) + f(x_1) & \text{if } x = x_0 \\ f(x) & \text{otherwise.} \end{cases}$$

Then $T \in \mathcal{B}(C_0(X))$ is bijective and $S_T(C_0(X)) = \mathbb{C}\varepsilon_{x_0}$.

For the first part we need the following, which follows immediately from a theorem of Gleason, Kahane and Żelazko.

Proposition

Let A be a Banach algebra with unit denoted e and B a commutative, semisimple unital Banach algebra. Let $T : A \to B$ be linear. Suppose that, for all invertible $a \in A$ we have that $T(a)$ is invertible. Then $a \mapsto T(e)^{-1}T(a)$ is multiplicative.

Proof of theorem (sketch)

- Reduce to case where X and Y are compact (i.e. $C_0(X) = C(X)$ and $C_0(Y) = C(Y)$ are unital).
- Show that (in unital case) T^{-1} maps invertibles to invertibles.
- Show that $T(1_X) = 1_Y$.
- Then, by the previous proposition, T^{-1} is multiplicative, and thus so is T.

Show that (in unital case) T^{-1} maps invertible to invertibles.

Let $T : C(X) \to C(Y)$ be a bounded, bijective semi-wcf-homomorphism. Assume that there exists a non-invertible $f \in C(X)$ such that $T(f)$ is invertible. Without loss of generality, we may assume that f is zero on an open subset U of X. Since X is perfect U is infinite. Now, there exist functions $g_n \in C(X)$, $(n \in \mathbb{N})$ each supported on the zero set of f such that (g_n) has no weakly-convergent subsequence. Then $fg_n = 0$ so we have

$$S_T(f, g_n) = T(f)T(g_n) - T(fg_n) = T(f)T(g_n).$$

Since T is a Banach space isomorphism and $T(f)$ is invertible it follows that $(S_T(f, g_n))_{n\in\mathbb{N}}$ has no weakly convergent subsequence, and so $T(f, \cdot)$ is not weakly compact. A **contradiction**.

Show that $T(1_X) = 1_Y$.

Let $T : C(X) \to C(Y)$ be a bounded, bijective semi-wcf-homomorphism. Then so is T^{-1}. Now, for $f \in C(Y)$,

$$\begin{aligned} S_{T^{-1}}(1_Y, f) &= T^{-1}(1_Y)T^{-1}(f) - T^{-1}(f) \\ &= (T^{-1}(1_Y) - 1_X)T^{-1}(f). \end{aligned}$$

Since T is onto we must have that multiplication by $(T^{-1}(1_Y) - 1_X)$ is a weakly compact linear map from $C(X)$ to itself. However, this would mean there was no infinite set on which $\left|(T^{-1}(1_Y) - 1_X)\right|$ was bounded below. Since X is perfect it follows that $T(1_X) = 1_Y$.

Open Questions

Since this work is in a very early stage, there are very many open questions.

- Under what circumstances are cf-homomorphisms automatically continuous?
- Under what circumstances are cf-homomorphism automatically multiplicative?
- For a given Banach algebra A what are the cf-homomorphisms from A to itself?
- What about compact failure of other identities say when $C_A(a, b) = ab - ba$ is compact? What can we say about A?